Volume of Ball and Hamming-type Bounds for Stiefel Manifold with Euclidean Distance

Renaud-Alexandre Pitaval and Olav Tirkkonen
Aalto University, Department of Communications and Networking, Espoo, Finland

***Abstract*—We compute the volume of the Stiefel manifold induced by the canonical embedding of the manifold as a surface in a Euclidean hypersphere and taking the corresponding Euclidean/chordal distance. Exploiting a power series expansion of the volume element, the volume of a small metric ball under the chordal distance is evaluated. Evaluating the volume of a metric ball is critical to derive Hamming-type bounds. Using a spherical embedding argument, we provide results generalizing previously known bounds on codes in the Grassmann manifold and the unitary group.**

I. Introduction

Unitary, Stiefel and Grassmann codes find application in space-time coding [1] and channel-aware precoding [2], [3] for multi-input multi-output (MIMO) communications. The complex Stiefel manifold $\mathcal{V}_{n,p}^{\mathbb{C}}$ is the space of $(n \times p)$-rectangular unitary matrices. The unitary group $\mathcal{U}_n = \mathcal{V}_{n,n}^{\mathbb{C}}$ and hypersphere $S^{2n-1} = \mathcal{V}_{n,1}^{\mathbb{C}}$ are specific cases of it. The Grassmann manifold $\mathcal{G}_{n,p}^{\mathbb{C}}$ is the space of eigenspaces spanned by the Stiefel matrices. In coding theory, the typical distances considered arise by treating those manifolds as Riemannian manifolds embedded in a Euclidean space and taking the natural Euclidean/chordal distance. To this extrinsic Euclidean distance corresponds an equivalent intrinsic Riemannian metric[1]. For the case of the Stiefel manifold, two non-equivalent Riemannian metrics are often considered [5], one realized from the space of rectangular unitary matrices $\mathcal{V}_{n,p}^{\mathbb{C}} \subset \mathbb{C}^{n\times p}$ and the other as a quotient space of the unitary group embedded in the space of square unitary matrices $\mathcal{V}_{n,p}^{\mathbb{C}} \cong \mathcal{U}_n/\mathcal{U}_{n-p} \subset \mathbb{C}^{n^2}$.

In the last decade, several works have been devoted to derive basic coding-theoretic results such as the Gilbert-Varshamov and Hamming bounds in these spaces [6]–[13]. The bounds estimate the relationship between the cardinality and the minimum distance of the code. The main difficulty in deriving such bounds is to evaluate the (normalized) volume of a small ball on the manifold. Asymptotic bounds ($n \to \infty$) on the rate/cardinality of Grassmannian codes were obtained in [6] employing an asymptotic evaluation of the volume of balls. In the unitary group, Hamming-type bounds on the minimal distance of the code were derived in [9] based on a simplification of the volume element that has to be integrated then relying on numerical evaluation. To derive further bounds on the minimal distance, inequalities between the geodesic distance induced by the quotient space structure and the chordal distance induced by the rectangular embedding were derived in [8], showing local (resp. global) equivalence of these two metrics for the Stiefel (resp. Grassmann) manifold. The volume of balls under the geodesic distance is then analyzed, relying partly on numerical evaluation, and the results are finally applied for the chordal distance using those inequalities. In [10], a closed-form expression on the volume of a small ball in Grassmannians under the chordal distance was derived. Finally, a power series expansion of the (unnormalized) volume of small ball valid for any Riemann manifold is leveraged in [11], [12]. This provides a powerful tool—in order to obtain a normalized volume expansion, it suffices to divide by the overall volume of the manifold.

However, it appears that in most of the literature, the volumes of the manifolds do not correspond to the natural metrics induced by the rectangular or unitary group embeddings. Indeed, the volume element is unique up to a non-vanishing scaling factor which is often dismissed. This is typically done for example when computing the Haar measure as this scaling factor can be absorbed in the overall normalization. Here, we are interested in bounds on codes in these manifolds, and the accurate normalization is of prime interest. A discussion and clarification of different conventional normalizations of the volume of the unitary group is provided in [14]. In [12] the volume of the complex Stiefel manifold is computed for the geodesic distance induced by the quotient geometry. In this paper, we address the problem when considering the typical chordal distance induced by the rectangular matrix embedding, which leads to another expression of the volume than the ones previously derived in [12] or [8].

A direct application of this volume estimate is to revisit and generalize the Hamming-type bounds of [8], [9]. As the distance we are working with is extrinsic and thus does not satisfy the triangle inequality with equality, there is some room for improvement of the standard Hamming bound following the same geometric argument used in [6], [9] for the Grassmann manifold and unitary group. We provide the corresponding result for the Stiefel manifold.

II. Preliminarities

A. Spaces

We consider the following Riemannian manifolds equipped with a chordal distance induced by their canonical embedding in a hypersphere.

- The $(d-1)$-*sphere* of radius r in $\mathbb{R}^d$:

$$\mathcal{S}^{d-1}(r) = \left\{ x \in \mathbb{R}^d \,|\, ||x|| = r \right\}.$$

[1]From the Nash embedding theorem, every Riemannian metric can be seen as induced by an appropriate Euclidean embedding [4].

For $r = 1$, we simply write $\mathcal{S}^{d-1}$. The $(d-1)$- and d-dimensional volume of $\mathcal{S}^{d-1}(r)$ are respectively

$$A_d(r) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1}, \quad V_d(r) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} r^d. \tag{1}$$

- The *unitary group*:

$$\mathcal{U}_n = \left\{ U \in \mathbb{C}^{n\times n} \quad | \quad U^H U = I_n \right\}.$$

We have $\dim \mathcal{U}_n = n^2$. The canonical embedding of $\mathcal{U}_n$ in the Euclidean space $(\mathbb{C}^{n\times n}, \langle\cdot,\cdot\rangle)$ where the inner product is typically defined as $\langle X, Y\rangle = \mathrm{Tr} X^H Y$ gives an isometric embedding in $\mathcal{S}^{2n^2-1}(\sqrt{n})$. The standard Euclidean/chordal distance considered is thus

$$d_c(U,V) = \|U - V\|_F = \left(4\sum \sin^2\theta_i/2\right)^{1/2} \tag{2}$$

where $\{e^{i\theta_i}\}$ are the eigenvalues of $U^H V$. Induced by this embedding, the tangent space at identity of the unitary group, $\mathfrak{u}(n)$, consists of skew-Hermitian n-by-n matrices. Without loss of generality assume that $U = I$ and $V = \exp(X) \in \mathcal{U}_n$ where $X \in \mathfrak{u}(n)$. The geodesic distance, the length between two points measured with the Riemannian metric, is given by

$$d_g(U,V) = \|X\|_F = \left(\sum \theta_i^2\right)^{1/2}. \tag{3}$$

- The complex *Stiefel manifold*, the space of orthonormal non-square matrices:

$$\mathcal{V}_{n,p}^{\mathbb{C}} = \left\{ Y \in \mathbb{C}^{n\times p} \quad | \quad Y^H Y = I_p \right\}.$$

We have $\dim \mathcal{V}_{n,p}^{\mathbb{C}} = (2np - p^2)$. The canonical embedding of $\mathcal{V}_{n,p}^{\mathbb{C}}$ in the Euclidean space $(\mathbb{C}^{n\times p}, \langle\cdot,\cdot\rangle)$, where $\langle X, Y\rangle = \mathrm{Tr} X^H Y$ gives an isometric embedding in $\mathcal{S}^{2np-1}(\sqrt{p})$ leading to the standard Euclidean/chordal distance

$$d_c(U,V) = \|U - V\|_F. \tag{4}$$

Induced by this embedding, the corresponding tangent direction at the identity $I_{n,p} \triangleq \begin{pmatrix} I_p \\ 0 \end{pmatrix}$ is of the form $X = \begin{pmatrix} A \\ B \end{pmatrix}$ where $A \in \mathfrak{u}(p)$, and $B \in \mathbb{C}^{(n-p)\times p}$. The corresponding geodesic distance given $U = I_{n,p}$ and $V = \exp(X)$ is then [5]

$$d_g(U,V) = \|X\|_F = \left(\|A\|_F^2 + \|B\|_F^2\right)^{1/2}. \tag{5}$$

Alternatively, the Stiefel manifold can also be treated as the quotient space $\mathcal{V}_{n,p}^{\mathbb{C}} \cong \frac{\mathcal{U}_n}{\mathcal{U}_{n-p}}$, where a point in $\mathcal{V}_{n,p}^{\mathbb{C}}$ is an equivalent class of unitary matrices. The Stiefel manifold inherits the geometry of $\mathcal{U}_n$ embedded in $(\mathbb{C}^{n\times n}, \langle\cdot,\cdot\rangle)$, and tangents at identity of $\mathcal{V}_{n,p}^{\mathbb{C}}$ are of the form $X_2 = \begin{pmatrix} A & -B^H \\ B & 0 \end{pmatrix}$ where $A \in \mathfrak{u}(p)$, and $B \in \mathbb{C}^{(n-p)\times p}$. With this non-equivalent geometry, the geodesic distance between $U = I_{n,n}$ and $V = \exp(X)$ is $d_{g2}(U,V) = \|X_2\|_F = \left(\|A\|_F^2 + 2\|B\|_F^2\right)^{1/2}$. The metric induced by this embedding, discussed in [12], is not considered in this paper.

- The complex *Grassmann manifold* $\mathcal{G}_{n,p}^{\mathbb{C}}$, with $p \le n/2$, is the quotient space of $\mathcal{V}_{n,p}^{\mathbb{C}}$ over $\mathcal{U}_p$: $\mathcal{G}_{n,p}^{\mathbb{C}} \cong \frac{\mathcal{V}_{n,p}^{\mathbb{C}}}{\mathcal{U}_p} \subset \mathbb{C}^{n\times p}$, which can also be expressed as $\mathcal{G}_{n,p}^{\mathbb{C}} \cong \frac{\mathcal{U}_n}{\mathcal{U}_p\mathcal{U}_{n-p}} \subset \mathbb{C}^{n^2}$. We have $\dim \mathcal{G}_{n,p}^{\mathbb{C}} = 2p(n-p)$. The tangents of $\mathcal{G}_{n,p}^{\mathbb{C}}$ at the identity are of the form $X_1 = \begin{pmatrix} 0 \\ B \end{pmatrix} \subset \mathbb{C}^{n\times p}$ or $X_2 = \begin{pmatrix} 0 & -B^H \\ B & 0 \end{pmatrix} \subset \mathbb{C}^{n^2}$ for the first and second quotient representation, respectively, with $B \in \mathbb{C}^{(n-p)\times p}$. It follows that the geodesic distances induced by the two embeddings differ only by a factor of 2. Usually the geodesic distance is defined, given $U = I$ and $V = \exp(X)$, as

$$d_g(U,V) = \|X_1\|_F = \frac{1}{\sqrt{2}}\|X_2\|_F = \left(\sum \theta_i^2\right)^{1/2}, \tag{6}$$

where $\{\theta_i\}$ are the singular values of B. The Grassmann manifold also affords an isometric spherical embedding in $\mathcal{S}^{n^2-2}(\sqrt{\frac{p(n-p)}{2n}})$ with the associated chordal distance:

$$d_c(U,V) = \frac{1}{\sqrt{2}}\|UU^H - VV^H\|_F = \left(\sum \sin^2\theta_i\right)^{1/2}. \tag{7}$$

In the following, $\mathcal{M}$ stands for $\mathcal{U}_n$, $\mathcal{V}_{n,p}^{\mathbb{C}}$, or $\mathcal{G}_{n,p}^{\mathbb{C}}$ embedded in $\mathcal{S}^{D-1}(R)$ where $D = 2n^2$, $2np$ or n^2-1, and $R = \sqrt{n}$, $\sqrt{p}$ or $\sqrt{\frac{p(n-p)}{2n}}$, respectively.

B. Code, Ball and Kissing Radius

A (N,δ)-code, $\mathcal{C} = \{C_1, \ldots, C_N\} \subset \mathcal{M}$ is a finite subset of N points in $\mathcal{M}$ with minimum distance δ.

We denote by $B_{C_i}(\gamma)$ the metric ball of radius γ with center at C_i, defined as

$$B_{C_i}(\gamma) = \{P \in \mathcal{M} \ : \quad d_c(P, C_i) \le \gamma\}. \tag{8}$$

The normalized volume of a ball is $\mu(B(\gamma)) = \frac{\text{vol } B(\gamma)}{\text{vol } \mathcal{M}}$, such that $\mu(\mathcal{M}) = 1$.

The kissing radius of the code $\mathcal{C}$ is defined as the maximum radius of non-overlapping metric balls centered around the codewords:

$$\varrho = \sup_{\substack{B_{C_l}(\gamma)\cap B_{C_k}(\gamma) = \emptyset \\ \forall (k,l)\ k\neq l}} \gamma. \tag{9}$$

III. Volume of Ball

A. Small Ball Approximation

There exists a power series expansion for the volume of small geodesic ball for any Riemannian manifold $\mathcal{M}$ of dimension d [15]. This was recently leveraged by Krishnamachari and Varanasi [12] to derive volume of small ball in the Stiefel manifold with the geodesic distance associated with the quotient representation.

Limiting this expansion to a single term leads to

$$\text{vol } B(r) = V_d(r)(1 + O(r^2)). \tag{10}$$

This can be intuitively understood as follows: in a small neighborhood the manifold looks alike a Euclidean space and can be approximated by its tangent space. It follows that the volume of a small ball in the manifold can be approximated by the volume of a ball of same radius in the tangent space:

$$\text{vol } B(r) = \text{vol } \{\exp(X) | X \in T\mathcal{M}, \|X\| \leq r\} \quad (11)$$
$$\approx \text{vol } \{X | X \in T\mathcal{M}, \|X\| \leq r\} \quad (12)$$

As $d_g = d_c + O(d_c^3)$ [16], [17], this result is also valid for the chordal distance arising from an isometric embedding in $\mathbb{R}^D$. We have

Proposition 1: The volume of metric balls in the d-dimensional Stiefel manifold $\mathcal{V}_{n,p}^{\mathbb{C}}$ with rectangular matrix induced metrics d_g or d_c is

$$\mu(B(r)) = \frac{V_d(r)}{\text{vol } \mathcal{V}_{n,p}^{\mathbb{C}}}(1 + O(r^2)), \quad (13)$$

where the overall volume of the Stiefel manifold is given by

$$\text{vol } \mathcal{V}_{n,p}^{\mathbb{C}} = \frac{2^{\frac{p(p+1)}{2}} \pi^{np}}{\Gamma_p(n)}, \quad (14)$$

the complex multivariate gamma function is

$$\Gamma_p(n) = \pi^{\frac{p(p-1)}{2}} \prod_{i=1}^{p} \Gamma(n - i + 1), \quad (15)$$

and $V_d(r)$ is given in (1).

A detailed volume computation is provided in the Appendix.

Remark 1: While for the general case $n \neq p$ the result above is different from the one in [12], the two results coincide for $n = p$ as expected. Other coefficients of the series expansion, addressed in [12], require computation of the curvature of the Stiefel manifold.

Remark 2: Applying the same approach to the Grassmann manifold, the result matches [10]: The volume of metric balls in the d-dimensional Grasmannian $(\mathcal{G}_{n,p}^{\mathbb{C}}, d_c)$ or $(\mathcal{G}_{n,p}^{\mathbb{C}}, d_g)$ is

$$\mu(B(r)) = \frac{V_d(r)}{\text{vol } \mathcal{G}_{n,p}^{\mathbb{C}}}(1 + O(r^2)) \quad (16)$$

where

$$\text{vol } \mathcal{G}_{n,p}^{\mathbb{C}} = \pi^{d/2} \prod_{i=1}^{p} \frac{\Gamma(p - i + 1)}{\Gamma(n - i + 1)} \quad (17)$$

Proof: This follows directly from the quotient geometry: $\text{vol } \mathcal{G}_{n,p}^{\mathbb{C}} = \frac{\text{vol } \mathcal{V}_{n,p}^{\mathbb{C}}}{\text{vol } \mathcal{U}_p}$; or alternatively and indirectly from $\frac{\text{vol } \mathcal{U}_n}{\text{vol } \mathcal{U}_p \text{vol } \mathcal{U}_{n-p}}$ and then multiplying by $2^{-\frac{d}{2}}$ due to the $2^{\frac{1}{2}}$ in the definition of the geodesic distance. ∎

Note that comparing to the Grassmannian case in [12], there is a difference by a factor of $2^{2p - \frac{n-p}{2}(n-p+1)}$. Due to local equivalence, the result is identical for the chordal and geodesic distances using the first order of the power expansion.

For the chordal distance, the result of [10] is stronger as it gives a strict equality for $r < 1$.

B. Special Case: $\mathcal{V}_{n,1}^{\mathbb{C}}$

Remark 3: The volume of metric balls in $(\mathcal{V}_{n,1}^{\mathbb{C}} \cong \mathcal{S}^d, d_c)$ with $d = 2n - 1$ is exactly

$$\mu(B(r)) = \begin{cases} \frac{1}{2} I_{r^2\left(1 - \frac{r^2}{4}\right)}\left(\frac{d}{2}, \frac{1}{2}\right) & \text{for } r \leq \sqrt{2} \\ 1 - \frac{1}{2} I_{(4-r^2)\left(1 - \frac{4-r^2}{4}\right)}\left(\frac{d}{2}, \frac{1}{2}\right) & \text{for } r \geq \sqrt{2} \end{cases}, \quad (18)$$

where $I_x(a, b)$ is the regularized incomplete beta function.

Proof: This follows directly from the area of a hyper-spherical cap, evaluated in [18]. ∎

C. Simulations

The approximated volumes (13) for the unitary group (with $p = n$) are compared with simulation in Fig. 1. For the Stiefel manifold (with $p \neq n$), the exact volume (18) for $p = 1$ and the approximated volume (13) for $p > 1$ are compared with simulation in Fig 2.

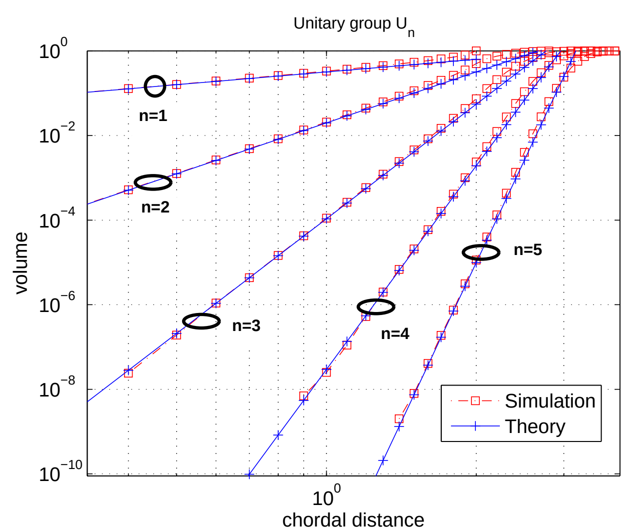

Fig. 1. Volume approximation of balls in the unitary group $\mathcal{U}_n$ compared to simulation.

IV. Kissing Radius and Hamming-type Bounds

For any (N, δ)-code, the standard Hamming bound reads

$$N\mu(B(\delta/2)) \leq 1. \quad (19)$$

The distances considered here are inherited from Euclidean embeddings. As the chordal distances are extrinsic to the considered curved spaces, they never satisfy the triangle inequality with equality. Obviously, it is possible to extend the radius $\frac{\delta}{2}$ so that the Hamming bound is still valid: for any (N, δ)-code,

$$N\mu(B(\varrho)) \leq 1 \quad (20)$$

where ϱ is the kissing radius of the code.

The difficulty in exploiting this bound is to find a relationship between the kissing radius and the minimum distance of the code. This is the object of this section.

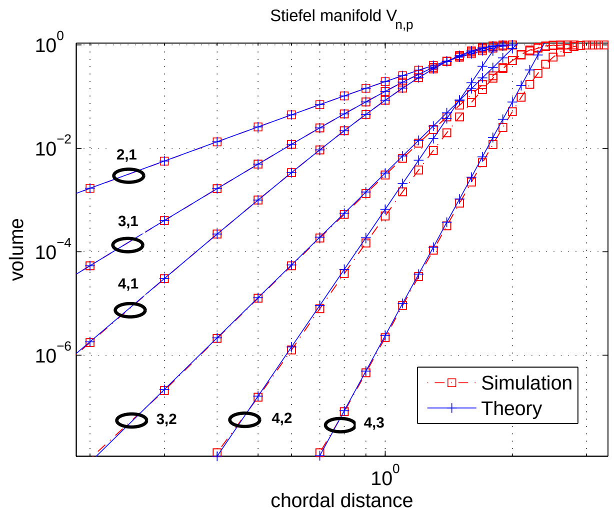

Fig. 2. Volume approximation of balls in the Stiefel manifold $\mathcal{V}^{\mathbb{C}}_{n,p}$ and exact volume of balls in $\mathcal{V}^{\mathbb{C}}_{n,1}$ compared to simulation.

A. Kissing Radius for Spherical Codes

Given an (N, δ)-spherical code on $\mathcal{S}^{D-1}(R)$, the midpoint on a geodesic of length δ is at distance ϱ_s from the extremities:

$$\varrho_s^2 = 2R^2 \left(1 - \sqrt{1 - \frac{\delta^2}{4R^2}}\right) \tag{21}$$

which can be inverted as

$$\delta^2 = 4\varrho_s^2 - \frac{\varrho_s^4}{R^2}. \tag{22}$$

B. Simple Bounds on Kissing Radius

As we consider a manifold isometrically embedded in a Euclidean sphere $\mathcal{S}^{D-1}(R)$, a (N, δ)-code in $\mathcal{M}$ is a (N, δ)-spherical code. Since balls of radius ϱ_s are non-overlapping on $\mathcal{S}^{D-1}(R)$, their inverse image on $\mathcal{M}$ are also non-overlapping, from which we can deduce that $\varrho_s \leq \varrho$.

Furthermore, assume that r_N fulfills:

$$\mu(B(r_N)) = \frac{1}{N} \tag{23}$$

Then we know that for every non-overlapping ball of radius r, we have $r \leq r_N$. Combining these facts gives

Lemma 1: Given a (N, δ)-code in $\mathcal{M}$, isometrically embedded in $\mathcal{S}^{D-1}(R)$, the kissing radius ϱ is bounded by

$$\varrho_s \leq \varrho \leq r_N, \tag{24}$$

where ϱ_s is given in (21) and r_N satisfies (23).

C. Hamming Bound

According to Lemma 1, we have the following Hamming bound:

Corollary 1: For any (N, δ)-code in $\mathcal{M}$ isometrically embedded in $\mathcal{S}^{D-1}(R)$, given ϱ_s defined in (21),

$$N\mu(B(\varrho_s)) \leq 1. \tag{25}$$

The geometric argument is similar to the one used in [6] to derive asymptotic bounds on Grassmannian codes and in [9] for the unitary group. A tighter bound for the Grassmannian case is provided in [13].

D. Hamming-type Bound on Minimal Distance

Combining (22) and Lemma 1, we have a Hamming-type bound for the minimum distance:

Proposition 2: Given a (N, δ)-code in $\mathcal{M}$, isometrically embedded in $\mathcal{S}^{D-1}(R)$, and a r_N satisfying (23), we have

$$\delta^2 \leq 4r_N^2 - \frac{r_N^4}{R^2}. \tag{26}$$

Applying this result to the unitary group leads to the bound [9, Theorem 2.4]. A tighter bound is provided for a small range of large distances by [9, Corollary 2.9]. While in [9] r_N is evaluated numerically, here r_N can be evaluated using Prop. 1. For the Grassmann manifold, a tighter bound for any distance can be found in [13]. For the Stiefel manifold, the result is new.

V. Conclusion

We have discussed the volume of metric balls and Hamming-type bounds on Stiefel codes with chordal distance. We compute the volume of the Stiefel manifold realized as a submanifold of an appropriate Euclidean space. We provide upper and lower bounds on kissing radius for packing of equal spheres in the Stiefel manifold. As a direct consequence, this result leads to a Hamming-type bound on the minimal distance of the code generalizing results known for the Grassmann manifold and the unitary group.

In the asymptotic regime, Hamming-type bounds on the Grassmannian code rate/cardinality [6] have been significantly improved by linear programming techniques [19]. Linear programming bounds for the asymptotic rate of complex Stiefel codes are also provided in [19]. In the non-asymptotic regime, fewer results are available. Linear programming bounds are provided for the real Grassmannian in [20] and for the unitary group in [21].

VI. Appendix: Proof of Proposition 1

Consider the Euclidean space of $m \times m$ complex matrices with its canonical inner product $(\mathbb{C}^{m\times m}, \langle\cdot,\cdot\rangle = \mathrm{Tr}\cdot^H\cdot)$. Given an $M \in \mathbb{C}^{m\times m}$, we have an infinitesimal squared distance in terms of the matrix differential,

$$ds^2 = \mathrm{Tr} dM^H dM = \sum_{j,k=1}^{m} \Re(dM_{jk})^2 + \Im(dM_{jk})^2 \tag{27}$$

Restricting the infinitesimal metric to the space of skew-hermitian matrices, with $A \in \mathfrak{u}(m)$, we get

$$ds_A^2 = -\mathrm{Tr} dA^2 = \sum_{j=1}^{m} |dA_{jj}|^2 + 2\sum_{j<k} |dA_{jk}|^2. \tag{28}$$

Note that a factor of 2 appears for off-diagonal terms as the matrix elements are coordinates of $\mathfrak{u}(m)$ in a non-normalized basis of $\mathfrak{u}(m)$. The corresponding metric tensor is diagonal with m ones and $m(m-1)$ twos. After computing the Jacobian determinant, the corresponding volume element is

$$d\omega_A = 2^{\frac{m(m-1)}{2}} \prod_{j=1}^{m} |dA_{jj}| \prod_{j<k}^{m} \Re(dA_{jk})\Im(dA_{jk}). \quad (29)$$

Given a unitary matrix $U \in \mathcal{U}_n$, by differentiating $U^H U = I$, we obtain

$$U^H dU + dU^H U = 0. \quad (30)$$

This shows that the differential one-form $U^H dU$ is skew-Hermitian. Due to the unitary invariance of the infinitesimal metric ds^2, its restriction to $\mathcal{U}_n$ can be expressed in terms of this global form, which is known as the Maurer–Cartan form of the unitary group. Then, the infinitesimal metric can be expressed as

$$ds_U^2 = -\mathrm{Tr}(U^H dU)^2 = \sum_{j=1}^{n} |(U^H dU)_{jj}|^2 + 2\sum_{j<k} |(U^H dU)_{jk}|^2 \quad (31)$$

and the volume form can be expressed as

$$d\omega_U = 2^{\frac{n(n-1)}{2}} d\nu_U \quad (32)$$

where

$$d\nu_U = \prod_{j=1}^{n} |(U^H dU)_{jj}| \prod_{j<k}^{n} \Re((U^H dU)_{jk})\Im((U^H dU)_{jk}). \quad (33)$$

The integration of $d\omega_U$, leading to the volume of the unitary group with the metric considered, was performed in [22] using Cayley's parametrization, with the result

$$\mathrm{vol}\ \mathcal{U}_n = \frac{2^{\frac{n(n+1)}{2}} \pi^{n^2}}{\Gamma_n(n)}. \quad (34)$$

Now let $Y \in \mathcal{V}_{n,p}^{\mathbb{C}}$ and $U \in \mathcal{U}_n$ such that $U^H Y = I_{n,p}$, i.e. the first p columns of $U = \begin{pmatrix} Y & Y^\perp \end{pmatrix}$ are the columns of Y. The differential form $U^H dY$ is "rectangular skew-hermitian", i.e. $U^H dY = \begin{pmatrix} Y^H dY \\ Y^{\perp H} dY \end{pmatrix}$ where $Y^H dY$ is p-by-p skew-Hermitian. As above, due to unitary invariance of the metric, the volume element for the Stiefel manifold can be expressed in terms of the global form $U^H dY$ and is given as

$$d\omega_Y = 2^{\frac{p(p-1)}{2}} d\nu_Y \quad (35)$$

where

$$d\nu_Y = \prod_{i=1}^{p} |(U^H dY)_{ii}| \prod_{k=1}^{p} \prod_{j=k+1}^{n} \Re((U^H dY)_{jk})\Im((U^H dY)_{jk}). \quad (36)$$

The volume element $d\nu_Y$ in (36) is the Haar measure normalized so that $\int_{\mathcal{V}_{n,p}^{\mathbb{C}}} d\nu_Y = \frac{2^p \pi^{np}}{\Gamma_p(n)}$, see e.g. [23] for the complex Stiefel manifold providing a generalization of real Stiefel result in [24] which employs the QR decomposition of Gaussian distributed matrices. Finally,

$$\mathrm{vol}\ \mathcal{V}_{n,p}^{\mathbb{C}} = \frac{2^{\frac{p(p+1)}{2}} \pi^{np}}{\Gamma_p(n)}. \quad (37)$$

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