On the Ergodic Mutual Information of Multiple Cluster Scattering MIMO Channels

Lu Wei, *Member, IEEE,* Zhong Zheng, *Student Member, IEEE,* Olav Tirkkonen, *Member, IEEE,* and Jyri Hämäläinen, *Member, IEEE*

***Abstract*—We consider multiple cluster scattering MIMO channels, where the effective channel equals the product of n complex Gaussian matrices. The considered channel model is typical for picocell propagation environments. For this channel model, we derive a closed-form lower bound of the ergodic mutual information. The derived lower bound is asymptotically tight, and is valid for an arbitrary number of clusters $n-1$ and an arbitrary number of antennas. The scaling law of the ergodic mutual information is found, generalizing the known result for the conventional MIMO channel model for which $n=1$.**

***Index Terms*—MIMO channel models; mutual information; picocell environment; product of Gaussian matrices.**

I. Introduction

To extend coverage and increase network capacity, picocells are increasingly used for dense teletraffic areas such as train stations, office buildings and airports. Picocellular propagation is affected by a wide range of mechanisms, among which an important characteristic is that the transmitted signal propagates through a sequence of scattering clusters (layers) until it reaches the destination. This multifold scattering channel model is typical in modeling, for example, indoor propagation between different floors [1, Chap. 13]. For multi-antenna transceivers, the end-to-end channel is modeled as a product of the multiple input multiple output (MIMO) channels of each cluster. This multiple cluster MIMO channel model has been considered in [2], and physical motivation for this channel model can be found in [3, Sec. 3].

Despite the importance of understanding information-theoretic quantities, such as the ergodic mutual information, of multiple cluster scattering MIMO channels, results in this direction are rather limited. Expressions for the ergodic mutual information of single-cluster and two-cluster scattering MIMO channels can be found in [4, Lemma 1] and [5, Eq. (9)], respectively, which were studied in the context of multi-hop MIMO relays. For an arbitrary number of clusters, the corresponding result is unavailable due to the absence of a computable formula for the singular value distribution of multiple cluster channels. For large number of transceivers, an asymptotic singular value distribution was proposed in [2]. This result, although of theoretical interest and formally valid for an arbitrary number of clusters, turns out to be too complicated for computational purposes.

To address this issue, we derive a computable ergodic mutual information lower bound for multiple cluster MIMO channels, which is valid for finite number of transceivers as opposed to the asymptotic result [2]. This lower bound is obtained by making use of Weyl's inequality as well as a recent result on the finite-dimensional eigenvalue statistics for products of complex Gaussian matrices. The derived lower bound becomes tight as the signal-to-noise ratio (SNR) approaches infinity, and numerical results show its usefulness for a wide range of SNR values. Based on the derived result, we gain physical insight into the characteristics of the considered channel model. In particular, the scaling behavior of the ergodic mutual information has been found. The derived scaling law is valid for any number of clusters, and generalizes the known result for the conventional MIMO channel.

II. System Model

Consider a MIMO system with a single source and destination, both equipped with K antennas. Information transmitted by the source is conveyed to the destination via $n-1$ successive clusters of scatterers. For analytical tractability, each cluster is assumed to have K scatterers.[1] The channels between non-consecutive clusters are ignored, and there is no direct link between the source and the destination.

The signal model of the MIMO system described above reads

$$\mathbf{y}=\sqrt{\frac{\gamma}{K^{n}}}\mathbf{P}_{n}\mathbf{x}+\mathbf{w}, \tag{1}$$

where the $K\times K$ matrix

$$\mathbf{P}_{n}=\mathbf{H}_{n}\cdots\mathbf{H}_{1}, \tag{2}$$

represents the effective scattering channel between the source and the destination, which equals the product of n independent $K\times K$ matrices. An illustration of the considered signal model can be found in Fig. 1, where each channel $\mathbf{H}_i$ is assumed to be an i.i.d Rayleigh fading channel i.e. the entries of $\mathbf{H}_i$ follow the standard complex Gaussian distribution and are independent of each other. The i.i.d Rayleigh channels require the so-called richly scattered physical environment, where there exist a large number of statistically independent reflected paths with random amplitudes [6, Chap. 7.3.8]. Thus, there needs to have a rich scattering environment creating $\mathbf{H}_i$, and a rich scattering environment creating $\mathbf{H}_{i+1}$. Between

The authors are with the Department of Communications and Networking, Aalto University, Finland (e-mail: {lu.wei, zhong.zheng, olav.tirkkonen, jyri.hamalainen}@aalto.fi). J. Hämäläinen is also with Ericsson Oulu R&D Center, Finland.

[1] We note that this assumption may not hold in practice. However, even with this assumption, certain insight and intuition on the problem of multiple cluster MIMO channels can be gained.

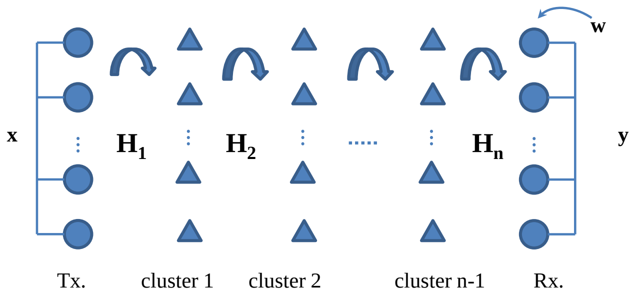

Fig. 1. Multiple cluster scattering MIMO channel with $n-1$ clusters. The number of antennas equals the number of scatterers per cluster.

these two environments, all scattering happens through the K scatterers in cluster i. These can be thought as K keyholes between these two environments. If there is only one scatterer in any of the clusters, the channel becomes a rank one keyhole channel as discussed in [7]. Referring to [1], the model (1) can be understood e.g. as the channel between floors in buildings, where inside each floor there is an i.i.d scattering environment, but between the floors there is restricted propagation through K scatterers. In line with the convention of [2, 4, 8], the effective channel $\mathbf{P}_n$ is normalized by $\sqrt{K^n}$ so that the total energy of the normalized channel[2] $\operatorname{tr}\left(\mathbb{E}\left[\mathbf{P}_n\mathbf{P}_n^\dagger\right]/K^n\right)=K$ does not depend on n. As a result, γ equals the average received SNR per antenna. In model (1), the noise $\mathbf{w}$ follows the standard complex Gaussian distribution with identity covariance matrix $\mathbb{E}[\mathbf{w}\mathbf{w}^\dagger]=\mathbf{I}_K$. Assuming that the effective channel $\mathbf{P}_n$ is only known to the destination, the covariance matrix of the input signal $\mathbf{x}$ is chosen to be $\mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]=\mathbf{I}_K$, where $\mathbf{x}$ follows the standard complex Gaussian distribution. Note that the signal model (1) is different from the multi-hop amplify-and-forward MIMO relay network as for the latter a noise term needs to be considered at each relay, leading to the signal model [8, Eq. (4)]. Assuming noiseless relays [5, 9], the two models become identical.

With the above notations, the mutual information in nats/s/Hz of the multiple cluster scattering MIMO channel reads

$$\mathcal{I}(\gamma)=\ln\det\left(\mathbf{I}_K+\frac{\gamma}{K^n}\mathbf{P}_n\mathbf{P}_n^\dagger\right)=\sum_{i=1}^{K}\ln\left(1+\frac{\gamma}{K^n}\lambda_i\right), \tag{3}$$

where $\ln(\cdot)$ is the natural logarithm and $\det(\cdot)$ denotes the matrix determinant. Here $0\leq\lambda_K\leq\ldots\leq\lambda_1<\infty$ are the squared singular values of the matrix $\mathbf{P}_n$, i.e. the eigenvalues of the Hermitian matrix $\mathbf{P}_n\mathbf{P}_n^\dagger$.

III. For the Ergodic Mutual Information Lower Bound

Evaluating the ergodic mutual information $\mathbb{E}[\mathcal{I}(\gamma)]$ requires the joint density of the ordered singular values or the arbitrary singular value density (average spectrum) of the channel matrix $\mathbf{P}_n$. There are no computable closed-form expressions for either of these for arbitrary n. However, utilizing an inequality between the singular values (real spectrum) and the eigenvalues (complex spectrum) of an instantaneous channel realization $\mathbf{P}_n$, combined with a recent result on the eigenvalue density for multiple cluster MIMO channels, a lower bound for $\mathbb{E}[\mathcal{I}(\gamma)]$ can be constructed.

Denote the squared absolute value of the eigenvalues of $\mathbf{P}_n$ as $0\leq|z_K|^2\leq\ldots\leq|z_1|^2<\infty$. Then the following inequality

$$\sum_{i=1}^{m}\varphi\left(|z_i|^2\right)\leq\sum_{i=1}^{m}\varphi(\lambda_i),\quad\forall m\in\{1,\ldots,K\}, \tag{4}$$

due to Weyl [10], holds for functions $\varphi(t)$ satisfying the conditions

- $\varphi(t)$ is an increasing function of $t>0$;
- $\varphi(\mathrm{e}^t)$ is a convex function of t;
- $\varphi(0)=\lim_{t\to 0}\varphi(t)=0$.

It can be easily verified that the function $\ln(1+t)$ fulfills the above conditions. Thus, a lower bound for the mutual information (3) is obtained as

$$\mathcal{I}_{\text{LB}}(\gamma)=\sum_{i=1}^{K}\ln\left(1+\frac{\gamma}{K^n}|z_i|^2\right) \tag{5}$$

$$\leq\sum_{i=1}^{K}\ln\left(1+\frac{\gamma}{K^n}\lambda_i\right)=\mathcal{I}(\gamma). \tag{6}$$

Now the task is to calculate the expected value of $\mathcal{I}_{\text{LB}}(\gamma)$. To this end we invoke a recent result on the joint density of $y_i=|z_i|^2$, $i=1,\ldots,K$ in [11]:

$$p(y_1,\ldots,y_K)=\operatorname{per}\left(\frac{y_i^{j-1}}{(\Gamma(j))^n}w_n\left(\sqrt{y_i}\right)\right), \tag{7}$$

where $\Gamma(\cdot)$ denotes the Gamma function, $0\leq y_K\leq\ldots\leq y_1<\infty$. In (7), the argument of the operator $\operatorname{per}(\cdot)$ is a $K\times K$ matrix i.e. $i,j=1,\ldots,K$. The operator $\operatorname{per}(\cdot)$ denotes a matrix permanent, i.e. for a $K\times K$ matrix $\mathbf{A}=(a_{i,j})$,

$$\operatorname{per}(\mathbf{A})=\sum_{\sigma\in S_K}\prod_{i=1}^{K}a_{i,\sigma(i)}, \tag{8}$$

where $\sigma=\sigma(1),\ldots,\sigma(K)$ is a permutation of the integers $1,\ldots,K$, and the sum is over all the $K!$ permutations S_K. In (7) the function

$$w_n\left(\sqrt{y_i}\right)=G_{0,n}^{n,0}\left(y_i\,\middle|\,\begin{matrix}-\\0,\ldots,0\end{matrix}\right) \tag{9}$$

stands for Meijer's G-function, the general form of which is given in (10) on top of the next page. Note that (7) is only applicable to the case when the number of antennas equals the number of scatterers per cluster. This is because each complex Gaussian matrix $\mathbf{H}_i$ is assumed to be a square matrix in [11]. Generalizing (7) to the case of unequal number of scatterers in each cluster seems difficult. Integrating the lower bound (5) over the density (7) in the domain $0\leq y_K\leq\ldots\leq y_1<\infty$, the lower bound of the ergodic mutual information is obtained. As indicated in [11], such integrations can be greatly simplified by making use of a property of matrix permanent for order statistics. Namely, it was proven in [12] that if a joint density

[2] $\operatorname{tr}(\cdot)$ denotes the matrix trace operation, and $(\cdot)^\dagger$ denotes the conjugate-transpose.

$$G_{p,q}^{m,n}\left(x\,\middle|\,\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j+z)\prod_{j=1}^{n}\Gamma\left(1-a_j-z\right)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-z\right)}x^{-z}\,\mathrm{d}z. \tag{10}$$

of ordered random variables $0\leq y_K\leq\ldots\leq y_1<\infty$ is written in the form $\mathrm{per}\left(h_j(y_i)\right),\ i,j=1,\ldots,K$, then the corresponding unordered random variables, denoted by x_j, $j=1,\ldots,K$, are independent of each other with densities $h_j(x)$, $j=1,\ldots,K$. Clearly, for the joint density (7) the resulting density functions of the unordered and independent random variables are

$$h_j(x)=\frac{x^{j-1}}{\left(\Gamma(j)\right)^n}w_n\left(\sqrt{x}\right),\quad j=1,\ldots,K. \tag{11}$$

Instead of dealing with random variables with non-trivial correlation as specified by a matrix permanent, we are now dealing with independent random variables. As such, the lower bound of the ergodic mutual information $\mathbb{E}[\mathcal{I}_{\mathrm{LB}}(\gamma)]$ can now be calculated as

$$\mathbb{E}\left[\mathcal{I}_{\mathrm{LB}}(\gamma)\right]=\sum_{j=1}^{K}\mathbb{E}\left[\ln\left(1+\frac{\gamma}{K^n}x_j\right)\right] \tag{12}$$

$$=\sum_{j=1}^{K}\int_0^\infty\ln\left(1+\frac{\gamma}{K^n}x\right)\frac{x^{j-1}}{\left(\Gamma(j)\right)^n}w_n\left(\sqrt{x}\right)\mathrm{d}x \tag{13}$$

$$=\sum_{j=1}^{K}\frac{1}{(\Gamma(j))^n}G_{2,n+2}^{n+2,1}\left(\frac{K^n}{\gamma}\,\middle|\,\begin{array}{c}0,1\\ 0,0,i,\cdots,i\end{array}\right), \tag{14}$$

where the last equality is obtained by [13, Eq. (21)] and using the fact [14, Eq. (8.4.6.5)]

$$\ln(1+x)=G_{2,2}^{1,2}\left(x\,\middle|\,\begin{array}{c}1,1\\ 1,0\end{array}\right). \tag{15}$$

Besides mutual information, the developed approximation framework is also applicable to performance analysis of MIMO multiple cluster scattering channels involving other linear statistics[3], such as the SNR distribution of space-time block coded transmission [15] and the outage probability of the MMSE receiver [16].

An important property of the derived ergodic mutual information lower bound (14) is its asymptotical tightness when the SNR goes to infinity. For $\gamma\to\infty$, the lower bound becomes exact

$$\mathcal{I}_{\mathrm{LB}}(\gamma)=\ln\left(\left(\frac{\gamma}{K^n}\right)^K\prod_{i=1}^{K}|z_i|^2\right) \tag{16}$$

$$=\ln\left(\left(\frac{\gamma}{K^n}\right)^K\prod_{i=1}^{K}\lambda_i\right)=\mathcal{I}(\gamma), \tag{17}$$

which is due to the fact that for any square matrix $\mathbf{P}_n$

$$\prod_{i=1}^{K}|z_i|^2=|\det(\mathbf{P}_n)|^2=\left|\det\left(\mathbf{P}_n\mathbf{P}_n^\dagger\right)\right|=\prod_{i=1}^{K}\lambda_i. \tag{18}$$

[3]The sum of functions of singular values of the form $\sum_{i=1}^{K}f(\lambda_i)$, with $f(\cdot)$ satisfying the conditions of Weyl's inequality.

IV. Ergodic Mutual Information Scaling Law

The scaling law for the ergodic mutual information at the high SNR regime can be understood using the derived results and the associated asymptotic tightness property. Specifically, for $\gamma\to\infty$, a simpler expression for the ergodic mutual information is obtained by integrating (17) over (11) to obtain

$$\mathbb{E}[\mathcal{I}(\gamma)]=nK\left(\sum_{i=1}^{K}\frac{1}{i}-c-1\right)+K\ln\left(\frac{\gamma}{K^n}\right), \tag{19}$$

where $c\approx0.5772$ is Euler's constant defined as

$$c=\lim_{K\to\infty}\left(\sum_{i=1}^{K}\frac{1}{i}-\ln(K)\right). \tag{20}$$

Thus, the per antenna ergodic mutual information scaling law for high SNR is obtained as

$$\mu_n=\lim_{K\to\infty}\frac{\mathbb{E}[\mathcal{I}(\gamma)]}{K}=\ln(\gamma)-n, \tag{21}$$

where we have used the definition of Euler's constant (20). The derived large SNR scaling law (21) indicates that the per antenna ergodic mutual information scales as $\ln(\gamma)-n$, and is a decreasing function of the number of clusters. Simulations performed in Section V show the usefulness of the proposed large SNR ergodic mutual information scaling law.

For the conventional MIMO channel model $n=1$, the finite-SNR ergodic mutual information scaling law, in our notations, reads [17, p. 591]

$$\ln(\gamma)-1+\frac{\sqrt{1+4\gamma}-1}{2\gamma}+2\tanh^{-1}\frac{1}{\sqrt{1+4\gamma}}, \tag{22}$$

where $\tanh^{-1}(\cdot)$ is the inverse hyperbolic tangent function. Since

$$\lim_{\gamma\to\infty}\frac{\sqrt{1+4\gamma}-1}{2\gamma}+2\tanh^{-1}\frac{1}{\sqrt{1+4\gamma}}=0, \tag{23}$$

we thus recover the result for $n=1$ in [17] for large SNR.

V. Numerical Results and Discussion

In this section, the derived lower bound (14) for the ergodic mutual information is compared to the ergodic mutual information obtained through Monte-Carlo simulations for various numbers of clusters $n-1$ and scatterers per cluster K. Each simulation curve is obtained by averaging over 10^6 independent channel realizations.

In Fig. 2 we consider a scenario of 3-cluster scattering MIMO channels i.e. $n=4$ with the number of scatterers per cluster being $K=2,\ 4,$ and 8. We plot the ergodic mutual information $\mathbb{E}[\mathcal{I}(\gamma)]$ in nats/s/Hz as a function of the received SNR in dB. It is seen that the ergodic mutual information increases as the number of scatterers (transceiver antennas) K increases. This behavior is in line with the conventional

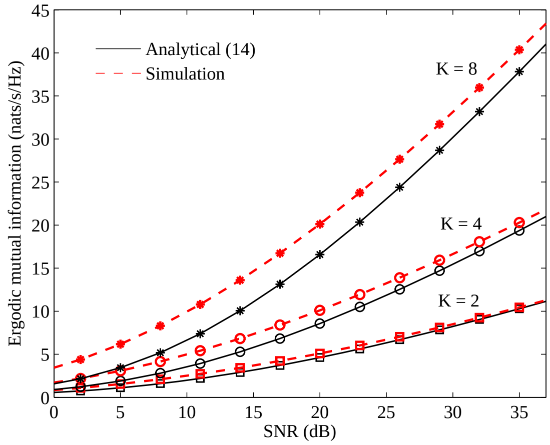

Fig. 2. Ergodic mutual information of multiple cluster scattering MIMO channels assuming 3 clusters ($n = 4$). In all cases, the number of antennas equals the number of scatterers per cluster.

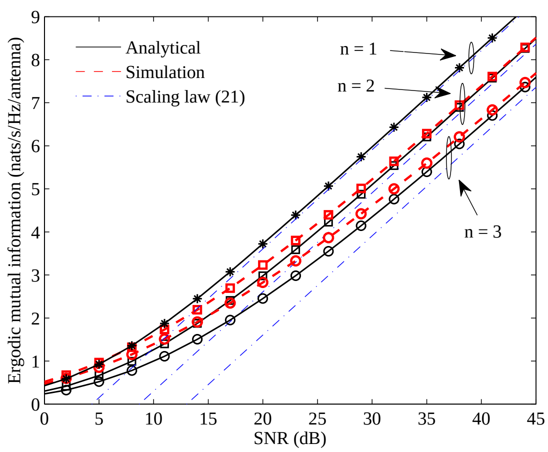

Fig. 3. Per antenna ergodic mutual information of multiple cluster scattering MIMO channels assuming $K = 8$ with different numbers of clusters $n-1$. In all cases, the number of antennas equals the number of scatterers per cluster. For $n = 1$ the analytical curve is obtained from the exact ergodic mutual information formula derived in [17]. For $n = 2, 3$ the analytical curves are obtained from the derived ergodic mutual information lower bound (14).

MIMO channel model $n = 1$, where increased spatial diversity (richness of the scattering) improves the ergodic mutual information [17].

In Fig. 3, a scenario involving a fixed number of scatterers per cluster $K = 8$ is considered with the number of clusters $n - 1$ being 1 and 2. In addition, the conventional MIMO channel model $n = 1$ has been considered. In order to verify the derived scaling law (21), we plot the per antenna ergodic mutual information in nats/s/Hz/antenna as a function of the received SNR in dB. It is observed that for a fixed SNR the ergodic mutual information decreases as the number of clusters increases. This observation is in agreement with the analytical scaling law (21). We also observe that the derived scaling law captures the behavior of the ergodic mutual information well at high SNR. As expected, we see from both figures that the derived lower bound (14) approaches the true value as the SNR increases.

VI. Conclusion

Multiple cluster scattering channel models are useful in modeling the picocell propagation environment. For such a channel model, we derived an analytical lower bound for the ergodic mutual information. The derived lower bound becomes tight as SNR goes to infinity, and is close to the true value for finite SNR. We also derived an ergodic mutual information scaling law for the high SNR regime, which is valid for arbitrary number of clusters. The derived scaling law implies that the ergodic mutual information of multiple cluster MIMO channels decreases as the number of clusters increases.

Acknowledgment

This work is supported by the Academy of Finland (Grants 257708 and 254299) and the Nokia Foundation. We thank R. Müller and G. Akemann for helpful discussions. We also wish to thank the reviewers and the editor for the constructive comments.

References

[1] S. R. Saunders and A. Aragón-Zavala, *Antennas and Propagation for Wireless Communication Systems.* 2nd Ed., New York: Wiley, 2007.

[2] R. Müller, "On the asymptotic eigenvalue distribution of concatenated vector-valued fading channels," *IEEE Trans. Inf. Theory,* vol. 48, no. 7, pp. 2086-2091, July 2002.

[3] J. B. Andersen and I. Kovacs, "Power distributions revisited," in *COST 273 TD (02) 004,* Jan. 2002.

[4] C. Zhong and T. Ratnarajah, "Ergodic sum rate analysis of Rayleigh product MIMO channels with linear MMSE receiver," in *Proc. IEEE Int. Symp. Inf. Theory,* July 2011.

[5] A. Firag, P. J. Smith, and M. R. McKay, "Capacity analysis of MIMO three product channels," in *Proc. Australian Commun. Theory Workshop,* Feb. 2010.

[6] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication.* Cambridge University Press, 2005.

[7] A. F. Molisch, "A generic model for MIMO wireless propagation channels in macro- and microcells," *IEEE Trans. Signal Process.,* vol. 52, no. 1, pp. 61-71, Jan. 2004.

[8] S. Borade, L. Zheng, and R. Gallager, "Amplify-and-forward in wireless relay networks: rate, diversity, and network size," *IEEE Trans. Inf. Theory,* vol. 53, no. 10, pp. 3302-3318, Oct. 2007.

[9] N. Fawaz, K. Zarifi, M. Debbah, and D. Gesbert, "Asymptotic capacity and optimal precoding in MIMO multi-hop relay networks," *IEEE Trans. Inf. Theory,* vol. 57, no. 4, pp. 2050-2069, Apr. 2011.

[10] H. Weyl, "Inequalities between the two kinds of eigenvalues of a linear transformation," in *Proc. Nat. Acad. Sci.,* vol. 35, pp. 408-411, July 1949.

[11] G. Akemann and E. Strahov, "Hole probabilities and overcrowding estimates for products of complex Gaussian matrices," *J. Stat. Phys.,* DOI 10.1007/s10955-013-0750-8, Apr. 2013.

[12] R. J. Vaughan and W. N. Venables, "Permanent expressions for order statistic densities," *J. Roy. Statist. Soc. Ser. B,* vol. 34, no. 2, pp. 308-310, 1972.

[13] V. S. Adamchik and O. I. Marichev, "The algorithm for calculating integrals of hypergeometric type functions and its realization in REDUCE system," in *Proc. Int. Conf. Symbolic and Algebraic Comput.,* pp. 212-224, 1990.

[14] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Volume 3: More special functions.* New York: Gordon and Breach Science, 1990.

[15] S. Sandhu and A. Paulraj, "Space-time block codes: a capacity perspective," *IEEE Commun. Lett.,* vol. 4, no. 12, pp. 384-386, Dec. 2000.

[16] K. R. Kumar, G. Caire, and A. L. Moustakas, "Asymptotic performance of linear receivers in MIMO fading channels," *IEEE Trans. Inf. Theory,* vol. 55, no. 10, pp. 4398-4418, Oct. 2009.

[17] İ. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecommun.,* vol. 10, no. 6, pp. 585-595, Nov. 1999.